On définit la suite: $U_n = \int_0^1 \frac{x^n}{\sqrt{1+x^2}}\,dx$, $n \in \mathbb{N}$

b) calculer $U_1 + U_3$; En déduire la valeur de U_3

$$U_1 + U_3 = \int_0^1 \frac{u}{\sqrt{1+u^2}}\,du + \int_0^1 \frac{u^3}{\sqrt{1+u^2}}\,dx$$

$$= \int_0^1 \left(\frac{u}{\sqrt{1+u^2}} + \frac{u^3}{\sqrt{1+u^2}} \right) du$$

$$= \int_0^1 \frac{u+u^3}{\sqrt{1+u^2}}\,du$$

$$= \int_0^1 \frac{u(1+u^2)}{\sqrt{1+u^2}}\,du \qquad \sqrt{1+u^2}^{\,2}$$

$$= \int_0^1 u.\sqrt{1+u^2}\,dx$$

$$(u'\sqrt{u} \rightarrow \frac{2}{3}\,u.\sqrt{u})$$

$$= \frac{1}{2}\int_0^1 \underbrace{2u}_{u'}.\sqrt{1+u^2}\,dx \qquad \sqrt{u}$$

$$= \frac{1}{2}\cdot\left[\frac{2}{3}.(1+u^2)\sqrt{1+u^2}\right]_0^1$$

$$= \frac{1}{3}\left[(1+u^2)\sqrt{1+u^2}\right]_0^1$$